# THE KANSAI-MARSHALL PROBE

故Marshall卿は先年，蒸気発生器伝熱管の傷を検出するための新しい渦電流プローブの原理を提案され，その理論解析を試みられた．この稿は，生前，技術システム研究所に数度にわたって送ってこられた卿の原稿を，未完の状態ではあるが，当研究所の卿に対する敬愛と感謝の気持を末永く残すために当研究所において編集したものである．

始めのうちは手書きの心覚えのようなものであったが，構想がまとまってきた頃に病を得られ，右腕が使えなくなり，パーソナルコンピューターを購入され，ギリシャ文字を打てるソフトウェアを組み込んで，多分左手で，論文をまとめにかかられたのである．

パーソナルコンピューターで作成された原稿は，Part Ⅰ，Part ⅡおよびPart Ⅲに分けて送ってこられたのであるが，病篤くなられ，「この積分は難しく，実行困難である」との文をもって，完成を見ることなく終わってしまった．

なお，卿が使われたソフトウェアは，数式表示には適しておらず，もとのままではわかり難いので，我々の見慣れたスタイルに書き改めた．また，何度も繰り返して検算を試みておられるため，文脈を追いにくいきらいがあったので，思い切って重複箇所を削除した．その他の点では，専ら原文の保存に努めた．

編集にあたっては，西村健主席研究員の協力を得た．記して謝意を表する．

1996年

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# THE KANSAI-MARSHALL PROBE

The Lord Marshall of Goring*

## 1. Introduction — Principles of the KM probe

The use of the simple bobbin coil in a probe to detect defects in Steam Generators is a well established technology which, on balance, produces a good compromise between "detectability power" and speed of operation. But it suffers from the disadvantage that it cannot reliably detect circumferential defects which are the most serious type of defects for the safety of the plant. There is therefore a need for a detection probe of a different type which retains the speed of the conventional bobbin coil but which can also detect circumferential defects.

This paper first describes the principles of such a device and gives a calculation of eddy-currents to demonstrate its theoretical capability. Next, this paper describes how these principles can be interpreted in a practical device which gives good experimental results. Lastly, an actual calculation of the signal from a circumferential defect is given.

We first review the basic principles upon which all eddy-current probes depend. If a uniform electromagnetic field of circular frequency "$\omega$" is present outside a metallic surface, then the field penetrates the metal to a certain distance called the "skin-depth", which depends upon both $\omega$ and $\sigma$. As we would expect, the penetration is largest for low frequencies and low conductivity.

If "$y$" measures the distance into the metal, then the amplitude of the field falls-off according to the law,

$$\exp\{-y(4\pi\sigma i\omega/c^2)^{1/2}\}$$

where the formula has been expressed in the Gaussian units familiar to physicists, "$c$" is the velocity of light and "$i$" is the square-root of "–1". This formula, as written, is inappropriate for direct use in calculations and it is therefore convenient to make two further definitions.

We first define the positive square-root of "$i$" as "$j$", hence,

$$j = i^{1/2} = (1+i)\,/\,2^{1/2}$$

We next define,

$$\Sigma^2 = 4\pi\sigma\omega\,/\,c^2.$$

and this permits us to rewrite the skin-depth formula as,

$$\exp(-yj\Sigma),$$

This demonstrates that the field falls off with an exponential form with parameter $\Sigma$ and also with a shift in phase which, in practice, is not usually very important. If readers of this paper prefer to work in units other than Gaussian units, they can use this last formula with $\Sigma$ expressed in their own choice of unit. The conventional

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eddy-current probe uses the basic results outlined above; the electrical conductivity is known from the material of the Steam Generator tubes and the frequency might typically be chosen so the field decays to one tenth of its amplitude at the outside of the tube. This is adequate to provide sufficient field throughout the metal. But alternating electromagnetic field in the metal implies the flow of electric currents in the metal; these currents are impeded by defects and cracks and this change is detected as a change in the impedance of the bobbin coil. This technology is therefore simple in principle, but more complicated in actual interpretation because it needs skill and experience to interpret impedance changes into defect types and sizes. The only real problem is the inability of this technology to detect circumferential defects which are thin. Therefore we now examine why this is so and propose an improvement.

The designs of bobbin coils vary one to the other but all of them involve a simple coil supplemented with ferrites placed to enhance the signal by creating some suitable concentration of field. All these devices retain cylindrical symmetry and therefore in all of them the currents flow in the circular path of the coil windings. By the fundamental law of action and reaction the currents induced in the material of the tube also flow in a circular direction. Thus if we use the usual cylindrical co-ordinates, $r$, $\theta$, $z$, then the currents flow in the $\theta$-direction both in the coil and in the metal of the tube.

The bobbin coil is dominated by large currents in the $\theta$-direction and large magnetic fields in the $z$-direction. Unfortunately a current in the $\theta$-direction does not intercept a circumferential defect unless that defect has some significant thickness. We conclude that the bobbin coil cannot detect an infinitely thin circumferential defect.

WE MUST BREAK THE CYLINDRICAL SYMMETRY

This is the crucial idea which will enable us to detect circumferential defects. Of course there is always one $\theta$-symmetry retained because any quantity evaluated at $\theta + 2\pi$ must be the same at $\theta$. Therefore we must produce a field varying like $\cos n\theta$ or $\sin n\theta$ where "$n$" is an integer not zero.

These thoughts lead to the basic idea of this new probe; to use ferrites to bend the field of a bobbin coil into the radial direction so that the end poles of the ferrites face outwards to the tube walls and are made to get as close to those walls as practically possible; to do this at the top and bottom of the coil so that the coil retains axial symmetry; and then alternate this arrangement with one omitting the radial poles as we go around the $\theta$ circle a distance of $2\pi R/N$ where $N$ is some integer. Looked at from above therefore the ferrite poles look like segments of a circle with every alternative segment empty. The number of ferrite segments on the central core is a matter of judgement; there could be 2, 3, 4, 5, 6, 7, 8, ... .

So far we have described how to break the cylindrical symmetry but the idea needs an additional development. For simplicity let us suppose we have chosen the "8-pole" version of the probe; then the initiating magnetic field will be a superposition of fields with $n = 0$, $n = 8$, $n = 16$, $n = 24$ ... in terms like $\cos n\theta$ and $\sin n\theta$. We can guess intuitively that only the first two of these will matter in practice. The term $n = 0$ will act exactly like a bobbin coil, will produce no axial current and will not "see" circumferential defects ; nevertheless it should be just as effective for other defects. The terms with $n = 8$ will have an axial current, will detect circumferential defects and will be "scattered" into terms like $n = 7$ or $n = 6$ because we assume the defect is only partially around the circumference. But terms like $n = 7$ and $n = 6$ cannot contribute to the self impedance of the coil, so although we have "detected" circumferential defects we have not "measured" them with only the present proposals.

We therefore add a further idea; we place central detector poles at the mid-point of the coil and wind detector coils to these detector poles so as to measure any flux diverted into them. These detector poles should be made of ferrite, should link back to the central core and should be made to get as close to the tube wall as practicable.

There are several other considerations which affect the detailed design of this probe, some are practical matters and some flow only from the difficult calculations. But we have now described the logic which leads to the present proposals. Therefore we now set about a calculation of the currents in the tube walls in order to verify the ideas described so far and calculate actual magnitudes of those currents and show they are large enough in principle for our purpose.

## 2. Calculation of current for a tube without defect

### 2.1 Maxwell's equations and potential functions

We first observe that the tube wall has a thickness, "*T*", which is small compared to the tube radius, "*R*", and that therefore we do not really need to use cylindrical co-ordinates. With negligible error we can use cartesian co-ordinates with the transformations

$$r \to y \qquad R\theta \to -x \qquad Z \to z$$

Notice the change of sign between "$R\theta$" and "$x$". This is required to ensure that "$x,y,z$" form a right-handed set. Note that the width of each pole piece is defined as $2\pi w/N$, so the total width of all $N$ poles is $2\pi w$. We shall later demonstrate that there is a practical advantage to choosing $w$ equal to $R/2$ so that the circumference is equally divided into "poles" and "non-poles".

We also notice that any variation in the $x$-direction must be periodic because a change in "$x$" of $2\pi R$ is a change in "$\theta$" of $2\pi$ and therefore brings us back to where we started. It follows that all functions of "$x$" vary like $\exp(ifx)$, where $f = n/R$ , $n$ being a positive or negative integer including zero. The primary coil is fed with a frequency "$\omega$", so all quantities vary with this frequency which is in the radio-frequency range in the form $\exp(i\omega t)$.

Maxwell's four equations, in Gaussian units are,

$$\nabla \times \boldsymbol{E} + \frac{i\omega}{c}\boldsymbol{B} = 0 \qquad \nabla \cdot \boldsymbol{B} = 0$$

$$\nabla \times \boldsymbol{H} - \frac{i\omega}{c}\boldsymbol{D} = \frac{4\pi}{c}\boldsymbol{J} \qquad \nabla \cdot \boldsymbol{D} = 4\pi\rho.$$

The tube walls have no magnetic permeability or dielectric constant, hence,

$$\boldsymbol{B} = \boldsymbol{H} \quad \text{and} \quad \boldsymbol{D} = \boldsymbol{E}.$$

From the second equation we therefore deduce,

$$\boldsymbol{H} = \nabla \times \boldsymbol{A},$$

where "$A$" is the vector potential. The first equation then yields,

$$\boldsymbol{E} + \frac{i\omega}{c}\boldsymbol{A} = -\nabla \cdot \phi,$$

where "$\phi$" is a scalar potential. We now choose a gauge so that

$$\nabla \cdot \boldsymbol{A} + \frac{i\omega}{c}\phi = 0.$$

Hence from the third equation we obtain,

$$\nabla^2 \boldsymbol{A} = -\frac{4\pi}{c}\boldsymbol{J}.$$

In deriving this equation, we have ignored the term which originates with the displacement current. The equivalent approximation in the gauge equation gives

$$\nabla \cdot \boldsymbol{A} = 0.$$

We can assume $\phi$ is zero, so

$$\boldsymbol{E} = -\frac{i\omega}{c}\boldsymbol{A}.$$

The current is given by,

$$\boldsymbol{J} = \sigma \boldsymbol{E},$$

so,

$$\nabla^2 \boldsymbol{A} = \frac{4\pi i \omega \sigma}{c^2}\boldsymbol{A} = i\Sigma^2 \boldsymbol{A}.$$

If now take the curl of this equation we get

$$\nabla^2 \boldsymbol{B} = i\Sigma^2 \boldsymbol{B},$$

and for simple situations it is possible to work directly with $B$, but for more complex problems it is necessary to work with $A$.

As the next step we Fourier analyse $A$ in the $x$ and $z$ directions to give,

$$\boldsymbol{A}[x,y,z] = \Sigma_f \int_{-\infty}^{+\infty} dk e^{ifx + ikz} \boldsymbol{A}[y],$$

which leads to,

$$\frac{\partial^2}{\partial y^2}\boldsymbol{A}[f,y,k] = (f^2 + k^2 + i\Sigma^2)\boldsymbol{A}[f,y,k].$$

It follows that $A[f,y,k]$ varies like,

$$\boldsymbol{A}[f,y,k] \approx \exp\{\pm y(f^2 + k^2 + i\Sigma^2)^{1/2}\}.$$

We notice that the expression in the brackets is complex and we therefore need to define it carefully. Obviously $(f^2 + k^2 + i\Sigma^2)$ is in the first quadrant of the complex plane so we define $(f^2 + k^2 + i\Sigma^2)^{1/2}$ to be in the first quadrant also. It follows that $-(f^2 + k^2 + i\Sigma^2)^{1/2}$ is in the third quadrant of the complex plane. We also note that for the usual frequencies $\Sigma^2$ is much larger than $(f^2 + k^2)$, so $(f^2 + k^2 + i\Sigma^2)$ is just to the right of the positive imaginary axis and $(f^2 + k^2 + i\Sigma^2)^{1/2}$ is therefore just below the +45 degree line in the complex plane. It follows that $-(f^2 + k^2 + i\Sigma^2)^{1/2}$ is just above the 225 degree line in the complex plane.

## 2.2 Magnetic field in the inner space of the tube

If we had chosen to work in terms of the magnetic field the last equation would be the exact analogue, namely,

$$\boldsymbol{B}[f,y,k] \approx \exp\{\pm y(f^2 + k^2 + i\Sigma^2)^{1/2}\}.$$

Between the poles and the tube inner surface $\Sigma$ is zero, because $\sigma = 0$. Then the magnetic field must decay in strength across the gap, G, and the incident field becomes

$$\boldsymbol{B}[f,k] \exp\{-(y + G)V\},$$

where $B[f, k]$ is the Fourier transform of the field at $y = -G$, and we have used the shorthand

$$V = (f^2 + k^2)^{1/2}.$$

Using this, we write the entire incident field as

$$B_Y = \Sigma_f \int dkB[f,k]\exp\{ifx + ikz - V(y+G)\}.$$

We can deduce the other components of the field by substituting $B_Y$ into the Maxwell equations. In total therefore, the initial wave is, for $-G < y < 0$

$$[B_X, B_Y, B_Z] = \Sigma_f \int dk[-if/V, 1, -ik/V]\, \boldsymbol{B}[f,k]\, \exp\{ifx + ikz - V(y+G)\}.$$

When this incident field reaches the tube wall, it generates a reflected wave which has some amplitude, which we shall name $\alpha[f,k]$, and decays in the opposite direction and originates at the inner tube wall, i.e. at $y = 0$. It therefore looks like $\exp(+Vy)$. Hence by analogy to the previous formula, the reflected wave is, for $-G < y < 0$

$$[B_X, B_Y, B_Z] = \Sigma_f \int dk[if/V, 1, ik/V]\alpha[f,k]\exp(ifx + ikz + Vy).$$

Hence the total field, between the pole pieces and the tube wall is, for $-G < y < 0$

$$[B_X, B_Y, B_Z] = \Sigma_f \int dke^{ifx+ikz}\{B[f,k]e^{-V(y+G)}[\frac{-if}{V}, 1, \frac{-ik}{V}] + a[f,k]e^{Vy}[\frac{if}{V}, 1, \frac{ik}{V}]\}.$$

## 2.3 Current inside the tube wall

Part of the incident wave enters the tube wall and then continues to decay according to the rules applicable to a conducting medium. We have discussed these rules before and remember that the wave must behave like $\exp\{\pm y(f^2 + k^2 + i\Sigma^2)^{1/2}\}$ depending upon whether the wave decays with decreasing $y$ or increasing $y$. The amplitude of the decreasing wave we call $\beta[f, k]$, and the amplitude of the increasing wave {which is reflected off the outer wall of the tube} we call $\gamma[f, k]$. The total wave belonging to this Fourier transform is therefore

$$B_Y[f,k] = \beta \exp(-yU) + \gamma \exp(yU),$$

where we have used the shorthand

$$U = (f^2 + k^2 + i\Sigma^2)^{1/2}.$$

Once again we substitute this into Maxwell's equations to obtain in total, INSIDE THE TUBE WALL, for $0 < y < T$

$$[B_X, B_Y, B_Z] = \Sigma_f \int dke^{ifx+ikz}\{\beta[f,k]\, e^{-Uy}\, [\frac{-ifU}{V^2}, 1, \frac{-ikU}{V^2}] + \gamma[f,k]\, e^{Uy}\, [\frac{ifU}{V^2}, 1, \frac{ikU}{V^2}]\}.$$

Finally, outside the tube where $y$ is larger than $T$, we have a decaying wave in free space with an amplitude we name $\varepsilon[f,k]$. Hence, OUTSIDE THE TUBE, for $T < y$

$$[B_X, B_Y, B_Z] = \Sigma_f \int dke^{ifx+ikz}\varepsilon[f,k]e^{-V(y-T)}[-if/V, 1, -ik/V].$$

We now have a series of expressions involving four unknowns $\{\alpha, \beta, \gamma$ and $\varepsilon\}$ for each choice of $f$ and $k$. We determine these unknowns by matching each solution across the boundaries at $y = 0$ and $y = T$. At $y = 0$ we get

$$B[f,k]\exp(-VG)\,[-if/V, 1, -ik/V] - \alpha[f,k]\,[if/V, 1, ik/V]$$
$$= \beta[f,k]\,[-ifU/V^2, 1, -ikU/V^2] + \gamma[f,k]\,[ifU/V^2, 1, ikU/V^2].$$

This gives three equations,

$$\frac{-if}{V}B[f,k]e^{-VG}+\frac{if}{V}\alpha=\frac{-ifU}{V^2}\beta+\frac{ifu}{V^2}\gamma,$$

$$B[f,k]e^{-VG}+\alpha=\beta+\gamma,$$

$$\frac{-ik}{V}B[f,k]e^{-VG}+\frac{ik}{V}\alpha=\frac{-ikU\beta}{V^2}+\frac{ikU\gamma}{V^2}.$$

The third equation is redundant.

Eliminating $\alpha$, we obtain

$$(V+U)\beta+(V-U)\gamma=2VBe^{-VG}.$$

Similarly from the boundary conditions at $y=T$ we obtain

$$(V-U)\beta\, e^{-UT}+(V+U)\gamma\, e^{UT}=0.$$

The two equations give $\beta$ and $\gamma$ separately as,

$$\beta=2VBe^{-VG}\frac{(V+U)e^{UT}}{(V+U)^2e^{UT}-(V-U)^2e^{-UT}},$$

$$\gamma=2VBe^{-VG}\frac{-(V-U)e^{-UT}}{(V+U)^2e^{UT}-(V-U)^2e^{-UT}}.$$

Consequently, at $y=0$,

$$B_Y=2VBe^{-VG}\frac{(V+U)e^{UT}-(V-U)\,e^{-UT}}{(V+U)^2e^{UT}-(V-U)^2e^{-UT}},$$

and at $y=T$,

$$B_Y=2VBe^{-VG}\frac{2U}{(V+U)^2e^{UT}-(V-U)^2e^{-UT}}.$$

Similarly, at $y=0$,

$$[B_X,0,B_Z]=2VBe^{-VG}[f,0,k]\frac{-iU}{V^2}\,\frac{(V+U)e^{UT}-(V-U)e^{-UT}}{(V+U)^2e^{UT}-(V-U)^2e^{-UT}},$$

and at $y=T$,

$$[B_X,0,B_Z]=2VBe^{-VG}[f,0,k]\frac{-iU}{V^2}\,\frac{2U}{(V+U)^2e^{UT}-(V-U)^2e^{-UT}}.$$

Substituting the above derived results in Maxwell's equations gives

$$\boldsymbol{J}=2\frac{i\sigma\omega}{cV}Be^{-VG}\frac{(V+U)e^{-U(y-T)}-(V-U)e^{U(y-T)}}{(V+U)^2e^{UT}-(V-U)^2e^{-UT}}[ik,0,-if],$$

so at $y=0$,

$$\boldsymbol{J}_{inside}=2\frac{i\sigma\omega}{cV}Be^{-VG}\frac{(V+U)e^{UT}-(V-U)e^{-UT}}{(V+U)^2e^{UT}-(V-U)^2e^{-UT}}[ik,0,-if],$$

and at $y=T$,

$$\boldsymbol{J}_{outside}=2\frac{i\sigma\omega}{cV}Be^{-VG}\frac{2U}{(V+U)^2e^{UT}-(V-U)^2e^{-UT}}[ik,0,-if].$$

## 2.4 Pattern of the pole pieces

We notice that the $z$-current is proportional to $f$ so it vanishes if we have cylindrical symmetry. Of course the current is also proportional to $\sigma$ and increases with the frequency.

This calculation cannot be taken any further until we have derived an expression for $B$, which is shorthand for $B[f,k]$; so we now turn to that task.

$B[f,k]$ is the Fourier transform of the $B_Y$ field emerging from the pole pieces. Hence,

$$B[f,k] = \frac{1}{4\pi^2 R}\int dxe^{-ifx}\int dze^{-ikz}B_0\{\text{Pattern of Pole Pieces}\}$$

In the $x$-direction the Pattern stretches

| | | | |
|---|---|---|---|
| from | $(R-w)\pi/N$ | to | $(R+w)\pi/N$ |
| from | $(R-w)\pi/N + 2\pi R/N$ | to | $(R+w)\pi/N + 2\pi R/N$ |
| from | $(R-w)\pi/N + 4\pi R/N$ | to | $(R+w)\pi/N + 4\pi R/N$ |
| .... | | | |

from $(R-w)\pi/N + 2(N-1)\pi R/N$ to $(R+w)\pi/N + 2(N-1)\pi R/N$

We now remember the observation made at the beginning that $f$ was restricted to values which obeyed the rule $f = n/R$ where $n$ is an integer. The $x$-integral then becomes

$$\int_{(R-w)\pi/N}^{(R+w)\pi/N} dxe^{-ifx}(1 + e^{-if2\pi R/N} + e^{-if4\pi R/N} + \cdots\cdots + e^{-if(N-1)2\pi R/N})$$

$$= e^{-in\pi/N}\frac{2R\sin(nw\pi/RN)}{n}\frac{1-e^{-i2n\pi}}{1-e^{-i2n\pi/N}}.$$

But, since $n$ is an integer the last term in the numerator is always zero and the integral vanishes unless the denominator is also zero. This occurs when $n$ is an integer times $N$ and then the fraction in the last expression becomes $N$. The special values of $f$ which satisfy this condition we call "$F$" so

$$F = Nm/R \quad \text{and} \quad n = Nm$$

and the integral becomes

$$(-1)^m\frac{2R\sin(mw\pi/R)}{m}.$$

When $F$ is zero, $m = 0$, and this is $2\pi w$.
When $F$ is $\pm N/R$, $m = \pm 1$, and this is $-2R\sin(\pi w/R)$.
When $F$ is $\pm 2N/R$, $m = \pm 2$, and this is $R\sin(2\pi w/R)$.

We want to concentrate the maximum effect into the first non-zero result i.e. into $F = \pm N/R$, i.e. $m = \pm 1$, and avoid overlap with the $\pm 2N/R$, i.e. $m = \pm 2$ effects. This last objective is best achieved by choosing $w$ so that the second mode has zero amplitude, i.e.

$$2\pi w/R = \pi \quad \text{i.e.} \quad w = R/2.$$

With this choice;

The zero mode has amplitude $R\pi$.
The first modes have amplitude $-2R$.
The second modes have amplitude zero.

Notice that each pole piece has width $2w\pi/N$, and the $N$ poles therefore have a total width of $2\pi w$. With our choice of $w$, this becomes $R\pi$ which is half of the total circumference. This seems intuitively correct .... to maximise the "breaking symmetry" by splitting the circumference half into poles and half into "non-poles".

We now turn to the $z$-integral;

$$\int dze^{-ikz} = \int_{-b}^{-a} dze^{-ikz}(-1) + \int_{a}^{b} dze^{-ikz}(+1) = \frac{2}{ik}\{\cos(ka) - \cos(kb)\}$$

Collecting these results together,

$$B[F,k] = \frac{1}{2\pi^2 m} B_0(-1)^m \sin(mw\pi/R)\frac{2}{ik}\{\cos(ka) - \cos(kb)\}$$

Finally we get the following formulae.

$$B_{Y,inside} = \Sigma_F \frac{1}{2\pi^2 m} B_0(-1)^m \sin(mw\pi/R) e^{iFx} \times \int dke^{ikz}\frac{2}{ik}\{\cos(ka) - \cos(kb)\}$$
$$\times 2Ve^{-VG}\frac{(V+U)e^{UT} - (V-U)e^{-UT}}{(V+U)^2 e^{UT} - (V-U)^2 e^{-UT}},$$

$$B_{Y,outside} = \Sigma_F \frac{1}{2\pi^2 m} B_0(-1)^m \sin(mw\pi/R) e^{iFx} \times \int dke^{ikz}\frac{2}{ik}\{\cos(ka) - \cos(kb)\}$$
$$\times 2Ve^{-VG}\frac{2U}{(V+U)^2 e^{UT} - (V-U)^2 e^{-UT}},$$

$$B_{X,Z,inside} = \Sigma_F \frac{1}{2\pi^2 m} B_0(-1)^m \sin(mw\pi/R) e^{iFx} \times \int dke^{ikz}\frac{2}{ik}\{\cos(ka) - \cos(kb)\}$$
$$\times 2Ve^{-VG}[F,0,k]\frac{-iU}{V^2}\frac{(V+U)e^{UT} - (V-U)e^{-UT}}{(V+U)^2 e^{UT} - (V-U)^2 e^{-UT}},$$

$$B_{X,Z,outside} = \Sigma_F \frac{1}{2\pi^2 m} B_0(-1)^m \sin(mw\pi/R) e^{iFx} \times \int dke^{ikz}\frac{2}{ik}\{\cos(ka) - \cos(kb)\}$$
$$\times 2Ve^{-VG}[F,0,k]\frac{-iU}{V^2}\frac{2U}{(V+U)^2 e^{UT} - (V-U)^2 e^{-UT}},$$

$$J_{X,Z,inside} = \Sigma_F \frac{1}{2\pi^2 m} B_0(-1)^m \sin(mw\pi/R) e^{iFx} \times \int dke^{ikz}\frac{2}{ik}\{\cos(ka) - \cos(kb)\}$$
$$\times \frac{i\sigma\omega}{c}\frac{2}{V}e^{-VG}\frac{(V+U)e^{UT} - (V-U)e^{-UT}}{(V+U)^2 e^{UT} - (V-U)^2 e^{-UT}}[ik,0,-iF],$$

$$J_{X,Z,outside} = \Sigma_F \frac{1}{2\pi^2 m} B_0(-1)^m \sin(mw\pi/R) e^{iFx} \times \int dke^{ikz}\frac{2}{ik}\{\cos(ka) - \cos(kb)\}$$
$$\times \frac{i\sigma\omega}{c}\frac{2}{V}e^{-VG}\frac{2U}{(V+U)^2 e^{UT} - (V-U)^2 e^{-UT}}[ik,0,-iF].$$

In all these formulae,

$$V = (F^2 + k^2)^{1/2} \text{ and } U = (F^2 + k^2 + i\Sigma^2)^{1/2}.$$

## 2.5 Evaluation of Fourier transform

These formulae can be evaluated only by going to the complex $k$-plane and it is therefore necessary to consider the general form of the expressions and identify singularities as a function of $k$. First we note that the first line of each formula is only concerned with the parameter "$F$". We can therefore set that aside for the moment. Furthermore we note that, if we choose $w = R/2$, the first line simplifies to

$$(B_0/4\pi)\exp(iFx) \quad \text{when } m = 0$$
$$(-B_0/2\pi^2)\exp(iFx) \quad \text{when } m = \pm 1$$
$$0 \quad \text{when } m = \pm 2$$

and is unimportant for other values of $m$.

Next we note that the second line always contains the expression,

$$\int dke^{ikz}\frac{2}{ik}\{\cos(ka)-\cos(kb)\} = \int dk\frac{1}{ik}\{-e^{ik[z+b]}+e^{ik[z+a]}+e^{ik[z-a]}-e^{ik\{z-b\}}\}.$$

We note that the expression in brackets goes to zero like $k^2$ at the origin and therefore we have no singularities at $k = 0$. For shorthand $Z$ can be used in succession to represent

$$[z+b],\ [z+a],\ [z-a] \text{ and } [z-b].$$

Finally we look at the complicated expressions in the second line. Because these involve $V$ and $U$ which are square-roots, we expect that it is necessary to introduce branch points and cuts in the complex plane which are connected to the zeros of these expressions. The zeros of $V$ occur when,

$$V = (k^2+F^2)^{1/2} = 0 \qquad \text{i.e.} \quad \text{when} \quad k = \pm\, iF.$$

and the zeros of $U$ occur when,

$$U = (k^2+F^2+i\Sigma^2)^{1/2} = 0 \qquad \text{i.e.} \quad \text{when} \quad k = \pm\, i(F^2+i\Sigma^2)^{1/2}.$$

But we note that, although it is not apparent at first glance, all the expressions are even in $U$ and therefore depend only on $U^2$ and therefore there is actually no singularity associated with $U$. This is a tremendous simplification. We are led to the conclusion that the expressions are analytic except for two branch points at $k = \pm iF$ and it is natural to make cuts along the imaginary $k$-axis from $+iF$ to $+i\infty$ and from $-iF$ to $-i\infty$.

Using these general formulae for $B_Y$ gives, for $z > 0$

$$B_{Y,inside} = (\int_A - \int_B)dke^{ikz}\frac{2Ve^{-VG}}{ik}\,\frac{(V+U)e^{UT}-(V-U)e^{-UT}}{(V+U)^2e^{UT}-(V-U)^2e^{-UT}}.$$

Along both contours, let $k = it$. Then,

$$V = (-t^2+F^2)^{1/2} = \pm\, i(t^2-F^2)^{1/2}.$$

At first sight the lower limit of the contour is simply $iF$, but we must remember that $F$ can be both positive and negative. Therefore we define,

$$s = \text{sign of } F,$$

and then the lower limit is $sF$ and must be positive. Furthermore,

$$U^2 = k^2+F^2+i\Sigma^2 = -t^2+F^2+i\Sigma^2,$$

and therefore as $t$ varies from $sF$ to $\infty$, $U^2$ varies from $i\Sigma^2$ to $i\Sigma^2-\infty$. Hence $U$ varies from $j\Sigma$ to $+(\varepsilon + i\infty)$, or alternatively from $-j\Sigma$ to $-(\varepsilon + i\infty)$ where $\varepsilon$ is small and positive. We shall choose the first choice for the variation of $U$. But then $U$ always has a positive real part. So exp($UT$) is much larger than exp($-UT$) over the important values of $t$. The formula then becomes,

$$\begin{aligned} B_{Y,inside} &= (\int_A - \int_B)dke^{ikZ}\frac{2Ve^{-VG}}{ik(V+U)} \\ &= \int_{5F}^{+\infty} idte^{-tZ}\frac{2i(t^2-F^2)^{1/2}\exp\{-iG(t^2-F^2)^{1/2}\}}{-t\{U+i(t^2-F^2)^{1/2}\}} \\ &\quad -\int_{5F}^{+\infty} idte^{-tZ}\frac{-2i(t^2-F^2)^{1/2}\exp\{-iG(t^2-F^2)^{1/2}\}}{-t\{U-i(t^2-F^2)^{1/2}\}} \end{aligned}$$

We are really only interested in this result for $F = 0$ because this is the dominating term. We now assume that $Z$ is bigger than the skin depth, $1/\Sigma$, and note that the term $\exp(-tZ)$ then cuts-off the integral for all large values of $t$. The denominators then become simply $U = j\Sigma$ and we get

$$B_{Y,inside,F=0} = \frac{2}{j\Sigma}\int_0^\infty dt(e^{-t(Z-iG)} + e^{-t(Z+iG)}) = \frac{4}{j\Sigma}\frac{Z}{Z^2+G^2}.$$

We now repeat the calculation for $Z < 0$. Then

$$B_{Y,inside} = (\int_E - \int_D)dke^{ikZ}\frac{2Ve^{-VG}}{ik}\frac{(V+U)e^{UT}-(V-U)e^{-UT}}{(V+U)^2e^{UT}-(V-U)^2e^{-UT}}.$$

As before, the final fraction reduces to $1/(j\Sigma)$. Let $k = -it$, then

$$B_{Y,inside,F=0} = \frac{1}{j\Sigma}\int_0^\infty \frac{-idt}{t}e^{tZ}(-2ite^{itG} - 2ite^{-itG}) = \frac{4Z}{j\Sigma(Z^2+G^2)}.$$

Therefore, for both $Z > 0$ and for $Z < 0$

$$B_{Y,inside,F=0} = \frac{4Z}{j\Sigma(Z^2+G^2)}.$$

We now remember that $Z$ stands for $z + b$, $z + a$, $z - a$ and $z - b$; and we should also remember the first line of our formulae so the full answer is,

$$B_{Y,inside,F=0} = \frac{2w}{\pi Rj\Sigma}\{-\frac{z+b}{(z+a)^2+G^2} + \frac{z+a}{(z+a)^2+G^2} + \frac{z-a}{(z-a)^2+G^2} - \frac{z-b}{(z-b)^2+G^2}\}$$

This expression is large opposite the pole pieces, is zero at $z = 0$, and falls away for $z$ large in either direction. Of course this is exactly what we expect.

In a similar way we can calculate the other components of $B$, both at the inside and outside the tube, but having illustrated one calculation there is no necessity to repeat similar exercises. But we do want to compare $J_{X,F=0}$ and $J_{Z,m=\pm 1}$ so we now examine them.

$$J_{X,Z,outside} = \Sigma_F\frac{1}{2\pi^2 m}B_0(-1)^m\sin(mw\pi/R)e^{iFx}\int dke^{-ikz}\frac{2}{ik}\{\cos(ka)-\cos(kb)\}$$
$$\times\frac{i\sigma\omega}{c}\frac{2}{V}e^{-VG}\frac{2U}{(V+U)^2e^{UT}-(V-U)^2e^{-UT}}[ik,0,-iF],$$
$$J_{X,outside,F=0} = \frac{B_0w}{2\pi R}\frac{i\sigma\omega}{c}\int dke^{ikz}\frac{2}{ik}\{\cos(ka)-\cos(kb)\}$$
$$\times\frac{2}{V}e^{-VG}\frac{2Uik}{(V+U)^2e^{UT}-(V-U)^2e^{-UT}}$$
$$= \frac{2B_0w}{\pi Rj\Sigma}\frac{i\sigma\omega}{c}e^{-j\Sigma T}\int dke^{ikz}\frac{e^{-VG}}{V}.$$

The final integral becomes,

$$(\int_A - \int_B)dke^{ikZ}\frac{e^{-VG}}{V} \quad \text{for} \quad Z > 0,$$
$$(\int_E - \int_D)dke^{ikZ}\frac{e^{-VG}}{V} \quad \text{for} \quad Z < 0.$$

In the first integral, let $k = it$. It then becomes,

$$I = \int_0^\infty idte^{-tZ}(\frac{e^{-itG}}{it} - \frac{e^{itG}}{-it}) = -\ln(Z+iG) - \ln(Z-iG) = -\ln(Z^2+G^2),$$

where "ln" is the natural logarithm. It can be proved that the same formula is valid for $Z < 0$. Hence,

$$J_{X,outside,F=0} = \frac{2B_0w}{\pi Rj\Sigma}\frac{i\sigma\omega}{c}e^{-j\Sigma T}\ln[\frac{\{(z-b)^2+G^2\}\{(z+b)^2+G^2\}}{\{(z-a)^2+G^2\}\{(z+a)^2+G^2\}}].$$

As a rough order of magnitude we note,

$$J_{X,outside,F=0} = \frac{2B_0 w}{\pi R j\Sigma}\ \frac{i\sigma\omega}{c} e^{-j\Sigma T} \times 4\ln(\frac{b}{a}).$$

The most difficult calculation is for $B_X$ and $B_Z$ so we outline it here. For variety we let $y$ be arbitrary. The start is,

$$\begin{aligned}[B_X, 0, B_Z] &= \frac{-iU}{V^2}[F,0,k]\{\beta[f,k]e^{-Uy} - \gamma[f,k]e^{Uy}\} \\ &= \frac{B_0(-1)^m \sin(\frac{mw\pi}{R})}{2\pi^2 m} e^{iFx} \int dk \frac{2}{ik}\{\cos(ka) - \cos(kb)\} \\ &\quad \times 2Ve^{-VG}[F,0,k]\frac{-iU}{V^2}\ \frac{(V+U)e^{UT-Uy} + (V-U)e^{-UT+Uy}}{(V+U)^2 e^{UT} - (V-U)^2 e^{-UT}}.\end{aligned}$$

Hence,

$$B_{Z,y,F=0} = \frac{B_0 w}{2\pi R}\int dk e^{ikZ} \frac{-2U}{V} e^{-VG} \frac{(V+U)e^{UT-Uy} + (V-U)e^{-UT+Uy}}{(V+U)^2 e^{UT} - (V-U)^2 e^{-UT}}.$$

As before, we can demonstrate that $U$ can be replaced by $j\Sigma$, to give,

$$\begin{aligned}B_{Z,y,F=0} &= -\frac{B_0 w}{\pi R j\Sigma} e^{-j\Sigma T}\int dk e^{ikZ} e^{-VG}(e^{j\Sigma(T-y)} + e^{-j\Sigma(T-y)}) \\ &\quad - \frac{B_0 w}{\pi R} e^{-j\Sigma T}\int dk e^{ikZ}\frac{1}{V} e^{-VG}(e^{j\Sigma(T-y)} - e^{-j\Sigma(T-y)}).\end{aligned}$$

The integrals are evaluated as before: for $Z > 0$

$$I_1 = \int dk e^{ikZ} e^{-VG} = \frac{2G}{Z^2 + G^2},$$

$$I_2 = \int dk e^{ikZ}\frac{e^{-VG}}{V} = \int_0^\infty i dt e^{-tZ}(\frac{e^{-itG}}{it} - \frac{e^{itG}}{-it}) = -\ln(Z^2 + G^2).$$

So finally,

$$\begin{aligned}B_{Z,y,F=0} &= -\frac{B_0 w}{\pi R j\Sigma} e^{-j\Sigma T}\frac{2G}{Z^2 + G^2}(e^{j\Sigma(T-y)} + e^{-j\Sigma(T-y)}) \\ &\quad + \frac{B_0 w}{\pi R} e^{-j\Sigma T}\ln(Z^2 + G^2)(e^{j\Sigma(T-y)} + e^{-j\Sigma(T-y)}).\end{aligned}$$

Only the second of these terms matter, so we get,

$$B_{Z,y,F=0} = \frac{B_0 w}{\pi R} e^{-j\Sigma T}\ln(Z^2 + G^2)(e^{j\Sigma(T-y)} + e^{-j\Sigma(T-y)}).$$

We remember that,

$$\boldsymbol{J} = \frac{c}{4\pi}\nabla \times \boldsymbol{B},$$

so,

$$J_X \approx \frac{c}{4\pi}\frac{\partial B_Z}{\partial y} = -j\Sigma\frac{B_0 wc}{4\pi^2 R} e^{-j\Sigma T}\ln(Z^2 + G^2)(e^{j\Sigma(T-y)} + e^{-j\Sigma(T-y)}),$$

and at $y = T$ this is,

$$J_{X,y=T} \approx -\frac{2i\sigma\omega B_0 w}{c\pi R j\Sigma} e^{-j\Sigma T}\ln(Z^2 + G^2).$$

which is the same result we derived previously.

We now calculate the $z$-current on the outside of the tube. We earlier derived,

$$J_{X,Z,outside} = \Sigma_F \frac{1}{2\pi^2 m} B_0(-1)^m \sin(mw\pi / R) e^{iFx} \times \int dk e^{ikz} \frac{2}{ik} \{\cos(ka) - \cos(kb)\} \frac{i\sigma\omega}{c} \times \frac{2}{V} e^{-VG} \frac{2U}{(V+U)^2 e^{UT} - (V-U)^2 e^{-UT}} [ik, 0, -iF],$$

so, for $m = \pm 1$,

$$J_{Z,outside,m=\pm 1} = iFB_0 \frac{\sin(w\pi / R)}{2\pi^2} e^{iFx} \frac{i\sigma\omega}{c} \frac{4}{ij\Sigma} e^{-j\Sigma T} \Sigma_Z I[Z],$$

where the integral is,

$$I[Z] = \int dt \frac{1}{kV} e^{ikZ - VG}.$$

For $Z > 0$, we can neglect $G$ to give

$$I[Z] = -2i \int_{sF}^{\infty} dt \frac{1}{t(t^2 - F^2)^{1/2}} e^{-tZ}.$$

This is not easy to evaluate exactly but is roughly of the form $\alpha e^{-\beta Z}$.
Let $t = sF\tau$, then,

$$I[Z] = \frac{-2i}{F} H[Z], \quad H[Z] = \int_1^{\infty} d\tau \frac{e^{-\tau sFZ}}{\tau(\tau^2 - 1)^{1/2}},$$

$$H[0] = \int_1^{\infty} \frac{d\tau}{\tau(\tau^2 - 1)^{1/2}} = \int_0^{\infty} d\theta \frac{\sinh\theta}{\cosh\theta \sinh\theta} = \int_1^{\infty} d\lambda \frac{2}{\lambda(\lambda + 1/\lambda)} = \int_1^{\infty} d\lambda \frac{2}{\lambda^2 + 1}.$$

Finally let $\lambda = \tan\phi$, so $d\lambda = d\phi/\cos^2\phi$

$$H[0] = \int_{\pi/4}^{\pi/2} d\phi 2 = \frac{\pi}{2}.$$

Similarly,

$$\int_0^{\infty} dZ H[Z] = \int_1^{\infty} \frac{d\tau}{sF\tau^2(\tau^2 - 1)^{1/2}} = \frac{1}{sF}.$$

Restoring the factor we set aside,

$$\alpha = \frac{-2i}{F} \frac{\pi}{2} = \frac{-i\pi}{F}, \quad \beta = \pi sF / 2.$$

Finally

$$I[Z] \approx -\frac{i\pi}{F} e^{-\pi sFZ/2}.$$

We now repeat the calculation for $Z < 0$.

$$I[Z] = \int dk \frac{1}{kV} e^{ikZ - VG}$$
$$= -\int_{-isF}^{-i\infty} dk \frac{1}{ki(-k^2 - F^2)^{1/2}} e^{ikZ} \exp\{-iG(-k^2 - F^2)^{1/2}\}$$
$$+ \int_{-isF}^{-i\infty} dk \frac{1}{-ki(-k^2 - F^2)^{1/2}} e^{ikZ} \{+iG(-k^2 - F^2)^{1/2}\}.$$

Let $k = -it$ so $dk = -idt$. Again we can neglect $G$ to give,

$$I[Z] = +2i\int_{sF}^{\infty} dt \frac{1}{t(t^2 - F^2)^{1/2}} e^{tZ}$$

We notice this has the opposite sign to the previous result, for $Z > 0$, so the combined result is,

$$I[Z] \approx -i\frac{\pi}{F}e^{-\pi sFZ} \quad \text{for} \quad Z > 0$$
$$I[Z] \approx +i\frac{\pi}{F}e^{+\pi sFZ} \quad \text{for} \quad Z < 0$$

And finally,

$$J_{Z,outside,m=\pm 1} = iFB_0 \frac{\sin(w\pi / R)}{2\pi^2} e^{iFx} \frac{i\sigma\omega}{c} \frac{4}{ij\Sigma} e^{-j\Sigma T}$$
$$\times \{-I[z+b] + I[z+a] + I[z-a] - I[z-b]\},$$

which we compare to,

$$J_{Z,outside,F=0,z=0} = 2B_0 \frac{w}{\pi Rj\Sigma} \frac{i\sigma\omega}{c} e^{-j\Sigma T} \times 4\ln\frac{b}{a}.$$

These are of the same magnitude and therefore *a priori* we can hope to detect circumferential defects. The next section considers this in detail.

## 3. Circumferential defect

## 3.1 Waves scattered by a circumferential defect

We have previously derived the result for a tube with no defect. It is

$$\{A_X; A_Y; A_Z; H_X; H_Y; H_Z\} = \Sigma_F \int dk e^{iFx + ikz - Uy} \beta_{Fk} \{-ik; 0; +iF; +iFU; F^2 + k^2; -ikU\}$$
$$+ \Sigma_F \int dk e^{iFx + ikz - Uy} \gamma_{Fk} \{-ik; 0; +iF; +iFU; F^2 + k^2; +ikU\},$$

$\sigma$ is the electrical conductivity in these units, and

$$(F^2 + k^2)\beta_{Fk} = -\frac{2VB(V+U)e^{-GV+TU}}{(V-U)^2 e^{-TU} - (V+U)^2 e^{+TU}}$$
$$(F^2 + k^2)\gamma_{Fk} = \frac{2VB(V-U)e^{-GV-TU}}{(V-U)^2 e^{-TU} - (V+U)^2 e^{+TU}}$$

where

$$B = (-1)^p \frac{w \sin(\pi wp / R)\{\cos(kz_2) - \cos(kz_1)\}B_0}{R(\pi wp / R)(-ik\pi)}.$$

Here $p$ is an integer which defines the values of $F$ by $F = 8p/R$, and $B_0$ is the magnetic field at the pole piece.

Each of these factors can be easily understood; $(-1)^p$ is a phase factor, $w/R$ is the ratio of the total width of the pole pieces to the total circumference, and the final factor, depending on $k$ is the Fourier transform of the pole pieces in the $z$-direction. The factor $\sin(\pi wp/R) / (\pi wp/R)$ is the Fourier transform of eight pole pieces in the $x$-direction.

Now insert a circumferential defect into the tube. We assume it to be infinitely thin and in the plane $z = Z$. We also define it to have a shape stretching from $T - D_X$ to $T$. We shall discuss possible shapes for the crack later in this calculation but we notice immediately that the shape $D_X$ must be periodic in $x$, and therefore $D_X$ and $D_{X+2\pi Rn}$, where $n$ is any integer, must be identical. Therefore $D_X$ can be expressed as a Fourier sum, i.e.

$$D_X = \Sigma_f D[f]e^{ifx}.$$

Where the coefficients, $D[f]$, are to be determined. This crack does not disturb the current flow in the $x$-direction but it stops the flow in the $z$-direction over the area of the crack. Therefore it introduces an additional result which we call the scattered field. At first we can only write this down in the plane of the crack, and only for $A_Z$. We therefore write it as,

$$A_Z[x,y,Z] = \Sigma_F \int dk e^{iFx+ikZ} \times \underline{CR}\{T - D_X < y < T\} \times (-iF)(\beta_{Fk}e^{-Uy} + \gamma_{Fk}e^{+Uy}),$$

and the crack function $\underline{CR}$ is, as indicated, unity for $y$ between $T - D_X$ and $T$ and zero elsewhere. Notice we have introduced the factor $iF$ with a minus sign so that this scattered field exactly cancels the original current flowing across the crack. We now suppose that the material of the tube is extended to infinity and this last equation describes the current flow from the half-plane below the plane $z = Z$ to the half-plane above. Then $A_Z$ is defined everywhere in this plane as a function of $x$ and $y$ and can be Fourier analysed as;

$$A_Z[x,y,Z] = \Sigma_f \int dq e^{ifx+iqy} A_Z[f,q,Z],$$

where

$$A_Z[f,g,Z] = \frac{1}{4\pi^2 R}\int_0^{2\pi R} d\zeta \int_{-\infty}^{+\infty} d\xi e^{-if\zeta - iq\xi} A_Z[\zeta,\xi,Z],$$

and the arguments of each integral have been changed to "$d\zeta$" and to "$d\xi$" appropriately and we have chosen these Greek variables as the Greek equivalents of $x$ and $y$.

Assembling these results, and performing the integrals over $\xi$, we now write down the final answer for $A_Z$ as,

$$A_Z[x,y,z] = \Sigma_F \int dk\beta_{Fk}\frac{F}{4\pi^2 R}e^{+ikZ}\Sigma_f e^{ifx}\int d\zeta e^{i(F-f)\zeta} \times \int dq e^{+iqy - W|z-Z|}\,\frac{e^{-(iq+U)\{T-D(\zeta)\}} - e^{-(iq+U)T}}{q - iU}$$
$$+ \Sigma_F \int dk\gamma_{Fk}\frac{F}{4\pi^2 R}e^{+ikZ}\Sigma_f e^{ifx}\int d\zeta e^{i(F-f)\zeta} \times \int dq e^{+iqy - W|z-Z|}\,\frac{e^{-(iq-U)\{T-D(\zeta)\}} - e^{-(iq-U)T}}{q + iU},$$

where,

$$W = (f^2 + q^2 + i\Sigma^2)^{1/2}$$

As the next step we must deduce the other components of the scattered field. We note that $A_Z$ behaves like,

$$\exp(+ifx + iqy - W|z - Z|)$$

We therefore recognise that all the field functions behave like that, including $H_Z$. But then $\nabla \cdot H$ has a discontinuity unless $H_Z$ is zero at $z = Z$. It follows that $H_Z$ is zero at $z = Z$ and therefore everywhere. We also remember that, under all conditions $\nabla \cdot A = 0$. This gives two equations,

$$ifA_X + iqA_Y = \pm WA_Z,$$
$$-iqA_X + ifA_Y = H_Z = 0.$$

Here we use the plus sign for $z \geq Z$, and the minus sign for $z < Z$.

The conclusion that $H_Z$ is zero everywhere in the scattered wave is so important that it deserves some discussion. First imagine the flow of current " through" the crack from one half plane to the other {this is the scattered wave which exactly compensates for the original flow in the position of the crack}. Then we can immediately determine the signs of $A_X$, $A_Y$, and $A_Z$ everywhere by looking at the obvious nature of the current flow — towards the crack in one half-plane and away from it in the other half plane —. But if we start from knowledge of $A_Z$ at the crack and choose a vector distance from the crack then we have knowledge of $H$ only through the vector project "$\nabla \times A$" and this gives no knowledge about $H$ in the $z$-direction. It follows that $H_Z$ must be zero. An alternative argument borrows a nomenclature from nuclear physics; $A_Z$ at the crack is a vector but $H_Z$ is a pseudo-vector, therefore $A_Z$ can determine $H_X$ and $H_Y$ but not $H_Z$. Therefore the latter must be zero.

The solution of these equations is,

$$A_X = \frac{\pm ifWA_Z}{-f^2 - q^2} = \frac{-isfWA_Z}{f^2 + q^2}$$
$$A_Y = \frac{\pm iqWA_Z}{-f^2 - q^2} = \frac{-isqWA_Z}{f^2 + q^2}.$$

Here we have introduced "$s$" to represent the "±" sign. We now easily write down the components of $H$ as;

$$H_X = iqA_Z + sWA_Y = iq(1 - \frac{s^2W^2}{f^2 + q^2})A_Z = \frac{q\Sigma^2 A_Z}{f^2 + q^2}$$
$$H_X = -sWA_X - ifA_Z = +if(\frac{W^2}{f^2 + q^2} - 1)A_Z = \frac{-f\Sigma^2 A_Z}{f^2 + q^2}.$$

Notice that $\nabla \cdot A$ and $\nabla \cdot H$ are zero as they should be. After somewhat lengthy effort of formula manipulation, we can derive the final answer

$$\underline{A,H}[x,y,z] = \Sigma_F \int dk(\beta_{Fk}e^{-UT} - \gamma_{Fk}e^{+UT})\frac{F}{4\pi^2 R}e^{ikZ}\Sigma_f e^{ifx}\int d\zeta e^{+i(F-f)\zeta}D[\zeta]$$
$$\times [i\int_{uper} dqe^{-iq(y-T)-Ws(z-Z)}\{\frac{-isfW}{f^2+q^2}; \frac{+isqW}{f^2+q^2}; 1; \frac{-q\Sigma^2}{f^2+q^2}; \frac{-f\Sigma^2}{f^2+q^2}; 0\}$$
$$-\pi De^{tf(y-T)-UT-js\Sigma(z-Z)} \times \{fst\Sigma; -ijs\Sigma; 0; \Sigma^2; -it\Sigma^2; 0\}].$$

To understand this, look at the exponentials. On the upper cut $q$ has a positive imaginary part, so $-iq$ has a positive real part, and $-iq(y - T)$ has a negative real part which gets more and more negative as $y$ moves away from $T$. This describes a wave decaying from the defect at $T$ towards the inner tube surface at $y = 0$. Similarly the pole term like $\exp\{tf(y - T)\}$ describes a wave decaying in the same direction. {When we look at waves beyond $T - D$ we shall discover waves decaying in the opposite sense.} Notice that this wave decays rapidly each side of the $z = Z$ plane and it does so through the rapid oscillation and decay $\exp\{\pm ig\psi s(z - Z)\}$; furthermore $\psi$ is near zero over the tip of the cut; therefore the decay either side of the $z = Z$ plane is sharp but not quite as sharp as the primary signal penetrating the tube in the first place.

The pole term is unexpected; it describes a wave strictly confined near the $z = Z$ plane but decaying very slowly towards the inner wall. Obviously this gives an excellent chance of detecting the defect unless some other unexpected phenomena turns up. But first we complete our calculation.

## 3.2 Signal magnification due to multiple scattering

Previously we evaluated the scattered wave for $y \le T - D$. Strictly we should next look at the intermediate range, $T - D \le y \le T$. But we want to evaluate these scattered waves at $y = T$ which is the boundary with the next region, $y \ge T$. Because the solution must be continuous we shall therefore look at this last region.

After similar procedure of calculation on the complex plane, it has been shown that the complete answer for $y$ at or beyond $T$ is the same as that derived above.

We have now obtained the formulae describing the scattered wave decaying towards the inner surface and that decaying towards the outer surface; but we have not yet taken account of those surfaces. We do that by noting that these formulae already give the general form of the waves and we therefore can immediately write down the waves reflected from the surfaces.

We have derived the important result that the crack gives a signal to the inside surface of the tube which is simply a concentrated burst of magnetic field centred on the plane of the crack. This concentrated burst is the sum of two terms; one is very concentrated indeed and is strong because it suffers little attenuation as it comes through the tube; this is the term like $e^{-js\Sigma(z-Z)}$ with an attenuation like $e^{-tfT}$. Furthermore this term is strongly amplified by the reflected beams from the inside and outside surfaces of the tube; in the formula this shows up in the term $(1 - e^{-2tfT})^{-1}$ which is very large because $e^{-2tfT}$ is only a little bit smaller than 1.

This magnification is so striking that it deserves a little explanation. The original signal from the crack arrives at the inner surface with amplitude proportional to $e^{-tfT}$; this stimulates a return wave which starts with amplitude $e^{-2ftT}$ and arrives at the outer surface with amplitude $e^{-2tfT}$; this in turn bounces inwards to give an amplitude at the inner surface of $e^{-3tfT}$; this in turn is bounced back and for to give another wave of amplitude $e^{-5tfT}$. This process is continued indefinitely to give the total sum

$$e^{-tfT} + e^{-3tfT} + e^{-5tfT} + e^{-7tfT} + e^{-9tfT} + \cdots\cdots = \frac{e^{-tfT}}{1 - e^{-2tfT}}.$$

To continue this calculation in principle we should now take Fourier transforms with respect to $z$, match to the field inside the tube and calculate how much of the emitted signal is gathered up by the detector. But our case is simpler because the emitted signal is so compressed in the $z$-direction that we can assume it is all captured if $Z$ is opposite the detector and nothing is captured otherwise. With this assumption the captured signal is zero, and we obtain finally the result,

$$SIGNAL = \Sigma_F \int \Sigma dk(\beta_{Fk} e^{-UT} + \gamma_{Fk} e^{+UT}) \frac{F}{4\pi^2 R} e^{ikZ}$$
$$\times \Sigma_f e^{+ifx} \int d\zeta e^{+i(F-f)\zeta} D[\zeta] \frac{-8jt\pi\Sigma e^{-tfT}}{1 - e^{-2tfT}} \times \underline{STEP}\{-\Delta \le Z \le +\Delta\},$$

where the step function indicates that the signal only exists when the crack is opposite the detector. We have previously derived an expression for $D[\zeta]$; it is,

$$D[\zeta] = \frac{Dw}{2R\pi^{1/2}}\{1 + \Sigma_{\phi>0} 2\cos\phi(\zeta - X)e^{-\phi^2 w^2/4}\},$$

where the crack is centred on the value "$X$", is assumed to have a Gaussian shape and "$w$" is a measure of its length.

This formula prompts us to look at the integral,

$$\int dk(\beta_{Fk} e^{-UT} + \gamma_{Fk} e^{+UT}) e^{+ikZ}$$
$$= \int dk e^{ikZ}(-4VUB)e^{-GV}(F^2 + k^2)\{(V-U)^2 e^{-TU} - (V+U)^2 e^{+TU}\}.$$

But this is exactly proportional to the total value of $A_Z$ on the outside of the tube, which is the same as the current on the outside of the tube before the crack disturbed it. This tells us that the signal arises directly opposite to the crack and is proportional to the original current at the cracks position. But we earlier proved that the current was exactly zero at the centre of the probe so the detector at that position is useless. We deduce we want the detector to be as high as possible almost touching the source pole pieces. This is also the geometry we want to protect the detector coils. We finally examine the dependence on $f$ to determine the variation of the signal around the circumference. Because this needs precise examination to get the symmetry conditions correct we proceed very carefully by identifying the contributions from positive and negative values of "$F$" and "$f$" separately, to give

$$SIGNAL = \Sigma_{F>0}\int \Sigma dk(\beta_{Fk}e^{-UT} + \gamma_{Fk}e^{+UT})\frac{-4jwDF\Sigma}{R\pi^{1/2}}e^{ikZ}$$
$$\times \Sigma_{f>0}\frac{e^{-|f|T}}{1-e^{-2|f|T}} \times \underline{STEP}\{-\Delta \le Z \le +\Delta\}$$
$$\times [e^{-(f-F)^2w^2/4}\{\cos(fx)\cos\{(f-F)X\} + \sin(fx)\sin\{(f-F)X\}\}$$
$$- e^{-(f+F)^2w^2/4}\{\cos(fx)\cos\{(f+F)X\} + \sin(fx)\sin\{(f+F)X\}\}].$$

which agrees with the previous definition.

In these expressions the terms involving $\exp\{-(f+F)^2w^2/4\}$ are always weak and the terms involving $\exp\{-(f-F)^2w^2/4\}$ are largest for $f$ close to $F$. But the factor $e^{-|f|T}/(1-e^{-2|f|T})$ is largest when $f$ is small. We may expect the exponential term to have the biggest effect for long cracks and the latter function to dominate for short cracks.

We now notice that the terms with $F$ equal to zero give zero signal, and we shall suppose that the width of the pole pieces, in total takes up half the circumference of the tube; then $w$ is $R/2$ and the complicated function $B$ listed at the beginning of this part and derived in the first part contains the factor $\sin(\pi wp/R)$. For this value of $w$, this is $\sin(\pi p/2)$, which is unity for $p = 1$ but zero for $p$ equal to 2 and –1 for $p = 3$. But we remember that $F = 8p/R$ so for $p = 3$, $F$ has the very large value of $24/R$. At this large value the terms in the formula can be ignored. We conclude we need give attention only to the term $F = 8/R$.

Next we remember that we do not observe the signal in the form of this Fourier transform; we actually measure eight signals which are the total integrated fields entering each segment of the detector poles. We shall label the signals to each of these segments $SIG.[f,n]$ where "$n$" runs from 0 to 7. Furthermore we can use a small computer to combine these signals in various ways. There are eight independent linear combinations of these signals which we define as

$$\underline{SIG.}[f,\Sigma,m] = \Sigma_n e^{+inm\pi/4}\underline{SIG.}[f,n],$$

where "$m$" can have values, 0, 1, 2, 3, 4, 5, 6, 7.

It is worth writing these coefficients down to give an insight to what each combination emphasises. We therefore make a table with "$n$" running horizontally and "$m$" running vertically.

| $m \backslash n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | $j$ | $i$ | $-j^*$ | $-1$ | $-j$ | $-i$ | $^*j$ |
| 2 | 1 | $i$ | $-1$ | $-i$ | 1 | $i$ | $-1$ | $-i$ |
| 3 | 1 | $-j^*$ | $-i$ | $j$ | $-1$ | $j^*$ | $i$ | $-j$ |
| 4 | 1 | $-1$ | 1 | $-1$ | 1 | $-1$ | 1 | $-1$ |
| 5 | 1 | $-j$ | $i$ | $j^*$ | $-1$ | $j$ | $-i$ | $-j^*$ |
| 6 | 1 | $-i$ | $-1$ | $i$ | 1 | $-i$ | $-1$ | $i$ |
| 7 | 1 | $j^*$ | $-i$ | $-j$ | $-1$ | $-j^*$ | $i$ | $j$ |

We have now set up the definitions we require and return to the evaluation of the signal. The final result is,

$$\underline{SIG.}[f,\Sigma,fR] = \int dk\{\beta_{Fk}e^{-UT} + \gamma_{Fk}e^{+UT}\}\frac{-16jwDF\Sigma}{R\pi^{1/2}}e^{ikZ}$$
$$\times \frac{e^{fT}}{1-e^{-fT}}\frac{\sin(f\pi R/8)}{f}\underline{STEP.}\{-\Delta \le Z \le +\Delta\}$$
$$\times (e^{-(F-f)^2w^2/4}e^{iX(F-f)} + e^{-(F+f)^2w^2/4}e^{iX(F+f)})\delta(m - fR).$$

The terms involving $\exp\{-(F+f)^2w^2/4\}$ are small and can be ignored in a first approximation. Remember that $F = 8/R$ and therefore, for a long crack, the terms that matter are $f = 7/R$ and $f = 6/R$. For a short crack the terms which matter are $f = 1/R$ and $f = 2/R$ because these have the weakest attenuation in the tube and are therefore magnified by the term $(1 - e^{-fT})$ in the denominator. Notice also that by measuring the phases of the various signals we can deduce the position of the crack around the circumference.

## 4. Detection of the scattered field

In the previous section, we did not give a precise calculation of how the magnetic field escapes the tube wall to reach the detector. This section sets out to correct this weakness and we begin by recalling some important results from our earlier work. We have derived the fundamental result that both $A$ and $H$ obeyed the law,

$$(\nabla^2 - i\Sigma^2)\{\boldsymbol{A}, \boldsymbol{H}\} = 0.$$

We already have some examples of this, which we recall as follows. The initial wave, inside the tube itself, behaves like

$$\exp(iFx + ikz + Vy) \quad \text{or} \quad \exp(iFx + ikz - Vy),$$

so

$$-F^2 - k^2 + V^2 = 0,$$
$$V = (F^2 + k^2)^{1/2}.$$

which is a result we obtained earlier.

Similarly, in the tube wall we have waves like

$$\exp(iFx + ikz + Uy) \quad \text{or} \quad \exp(iFx + ikz - Uy),$$

so,

$$-F^2 - k^2 + U^2 - i\Sigma^2 = 0,$$
$$U = (F^2 + k^2 + i\Sigma^2)^{1/2}.$$

The scattered waves looked like

$$\exp\{ifx + iqy + W(z - Z)\} \quad \text{or} \quad \exp\{ifx + iqy - W(z - Z)\},$$

and the bounced waves from the inside and outside surfaces of the tube wall looked exactly the same but with a suitable superposition of $+q$ and $-q$ waves. We have also noted that these bounced waves are of two types; one type is a superposition of waves over a range of $q$-values; the second type is the special choice of $q = \pm if$, which forces $W$ to the special value of $\pm j\Sigma$.

But if we want to examine how these waves get out of the tube wall, we need to Fourier analyse them into plane waves in the $x - z$ plane. This is a trivial step in the $x$-plane because the task is already done, each wave behaves like $\exp(ifx)$. But in the $z$-plane we must re-express $\exp\{\pm W(z - Z)\}$ as a Fourier superposition. This gives us so many variables that the calculation can be confusing.

We derived previously the scattered terms in the neighbourhood of $y = 0$ and in the neighbourhood of $y = T$. These waves must now be matched to the boundary conditions at $y = 0$ and $y = T$. The most important boundary conditions are that $A_y$ is zero on both surfaces;because this is an internal condition,internal to the tube wall, it must be obtained from bounced waves of the same form we have just written down. The other conditions are that

$H$ is continuous across both surfaces.The calculation is therefore done in two steps. In the first step we superimpose terms similar to those above but with arbitrary coefficients, $\mu_1$, $\mu_2$, $v_1$, $v_2$.

It may be shown that the integral term is small; of the order of $f/\Sigma$ times the pole term; and we can therefore ignore it. Hence the final results of this section are,

$$\{\boldsymbol{A}, \boldsymbol{H}\}[y \approx 0] = \Sigma_F \int dk(\beta_{Fk}e^{-UT} + \gamma_{Fk}e^{UT})\frac{F}{4\pi^2 R}e^{ikZ}\Sigma_f e^{ifx}\int d\zeta e^{i(F-f)\zeta}D[\zeta]$$
$$\times \pi e^{-js\Sigma(z-Z)}\{jst\Sigma;\ 0;\ 0;\ 0;\ -it\Sigma^2;\ 0\}\frac{4e^{tfT}}{-1+e^{2tfT}},$$

and

$$\{\boldsymbol{A}, \boldsymbol{H}\}[y \approx T] = \Sigma_F \int dk(\beta_{Fk}e^{-UT} + \gamma_{Fk}e^{UT})\frac{F}{4\pi^2 R}e^{ikZ}\Sigma_f e^{ifx}\int d\zeta e^{i(F-f)\zeta}D[\zeta]$$
$$\times \pi e^{-js\Sigma(z-Z)}\{jst\Sigma;\ 0;\ 0;\ 0;\ -it\Sigma^2;\ 0\}\frac{-2(1+e^{2ftT})}{1-e^{2tfT}}.$$

We notice, because of the presence of, $s$, that the current in the $x$-direction is opposite below and above the crack. Otherwise we need only notice the value of $H_Y$ at $y \approx 0$ and $y \approx T$. These we put in the form,

$$H_{Y=0} = \Sigma_F \int dk(\beta_{Fk}e^{-UT} + \gamma_{Fk}e^{UT})\frac{F}{4\pi^2 R}e^{ikZ}\Sigma_f e^{ifx}\int d\zeta e^{i(F-f)\zeta}D[\zeta]$$
$$\times \pi e^{-js\Sigma(z-Z)}(-it\Sigma^2)\frac{4e^{tfT}}{-1+e^{2tfT}},$$

and

$$H_{Y=T} = \Sigma_F \int dk(\beta_{Fk}e^{-UT} + \gamma_{Fk}e^{UT})\frac{F}{4\pi^2 R}e^{ikZ}\Sigma_f e^{ifx}\int d\zeta e^{i(F-f)\zeta}D[\zeta]$$
$$\times \pi e^{-js\Sigma(z-Z)}(-it\Sigma^2)\frac{2(1+e^{2ftT})}{-1+e^{2tfT}}.$$

The next step is to Fourier analyse $\pi e^{-js\Sigma(z-Z)}$ as,

$$\pi e^{-j\Sigma s(z-Z)} = \int_{-\infty}^{+\infty} dK e^{iK(z-Z)}\theta_K,$$

where,

$$\theta_K = (1/2\pi)\int dz e^{-iK(z-Z)}\pi e^{-js\Sigma(z-Z)} = \frac{j\Sigma}{K^2 + i\Sigma^2}.$$

Our results become,

$$H_{Y=0} = \Sigma_F \int dk(\beta_{Fk}e^{-UT} + \gamma_{Fk}e^{UT})\frac{F}{4\pi^2 R}e^{ikZ}\Sigma_f e^{ifx}\int d\zeta e^{i(F-f)\zeta}D[\zeta]$$
$$\times \int_{-\infty}^{+\infty} dK e^{iK(z-Z)}\theta_K(-it\Sigma^2)\frac{4e^{tfT}}{-1+e^{2tfT}},$$

and

$$H_{Y=T} = \Sigma_F \int dk(\beta_{Fk}e^{-UT} + \gamma_{Fk}e^{UT})\frac{F}{4\pi^2 R}e^{ikZ}\Sigma_f e^{ifx}\int d\zeta e^{i(F-f)\zeta}D[\zeta]$$
$$\times \int_{-\infty}^{+\infty} dK e^{iK(z-Z)}\theta_K(-it\Sigma^2)\frac{2(1+e^{2ftT})}{-1+e^{2tfT}}.$$

We now notice that both these expressions are of the form,

$$\Sigma_f \int dK e^{ifx + iK(z-Z)}$$

and we recognise that this form will extend beyond the tube surfaces and be reflected inside the tube itself, each part having the appropriate dependence on $y$. On the inside of the tube ($y < 0$) the wave looks like $\exp(+yN)$ where $N = (f^2 + K^2)^{1/2}$. On the outside ($y > T$) the wave looks like $\exp\{-(y - T)N\}$. Inside the tube wall ($0 < y < T$) we get $\exp(+yM)$ and $\exp(-yM)$ where $M = (f^2 + K^2 + i\Sigma^2)^{1/2}$.

In the previous calculation we eliminated $A_Y$ at the boundaries so we assume that any additional terms have $A_Y$ zero everywhere. We therefore assume $A_X$ and $A_Z$ are non-zero. Then it follows that,

$$
\begin{aligned}
H_X &= \partial A_Z / \partial y - \partial A_Y / \partial z \\
&= NA_Z = \{ifN / (f^2 + k^2)\}H_Y[y] \quad \text{for} \quad y < 0 \\
&= -NA_Z = \{-ifN / (f^2 + k^2)\}H_Y[y] \quad \text{for} \quad y > T \\
&= MA_Z = \{ifM / (f^2 + k^2)\}H_Y[y] \quad \text{for} \quad 0 < y < T\text{, growing} \\
&= -MA_Z = \{-ifM / (f^2 + k^2)\}H_Y[y] \quad \text{for} \quad 0 < y < T\text{, decaying}
\end{aligned}
$$

$$
\begin{aligned}
H_Z &= \partial A_Y / \partial x - \partial A_X / \partial y \\
&= -NA_X = \{iKN / (f^2 + k^2)\}H_Y[y] \quad \text{for} \quad y < 0 \\
&= NA_X = \{-iKN / (f^2 + k^2)\}H_Y[y] \quad \text{for} \quad y > T \\
&= -MA_X = \{iKM / (f^2 + k^2)\}H_Y[y] \quad \text{for} \quad 0 < y < T\text{, growing} \\
&= MA_X = \{-iKM / (f^2 + k^2)\}H_Y[y] \quad \text{for} \quad 0 < y < T\text{, decaying}
\end{aligned}
$$

Hence, <u>inside</u> the tube ($y < 0$),

$$H[y] = \{ifN / (f^2 + K^2);\ 1;\ iKN / (f^2 + K^2)\} \times a \times e^{Ny},$$

and, <u>outside</u> the tube ($y > T$),

$$H[y] = \{-ifN / (f^2 + K^2);\ 1;\ -iKN / (f^2 + K^2)\} \times d \times e^{N(y-T)},$$

and, <u>in the tube wall</u> itself ($0 < y < T$),

$$
\begin{aligned}
H[y] = &\{ifM / (f^2 + K^2);\ 1;\ iKM / (f^2 + K^2)\} \times b \times e^{My} \\
&+ \{-ifM / (f^2 + K^2);\ 1;\ -KM / (f^2 + K^2)\} \times c \times e^{-M(y-T)} + \mathit{SOURCE\ TERMS}.
\end{aligned}
$$

Matching the $x$-components across boundaries gives,

$$
\begin{aligned}
\frac{ifN}{f^2 + K^2} \times a &= \frac{ifM}{f^2 + K^2}(b - ce^{MT}), \\
\frac{-ifN}{f^2 + K^2} \times d &= \frac{ifM}{f^2 + K^2}(be^{MT} - c).
\end{aligned}
$$

Hence

$$a = (b - ce^{MT})M / N \qquad d = -(be^{MT} - c)M / N.$$

Matching $z$-components gives redundant equations. The $y$-components give,

$$
\begin{aligned}
a &= b + ce^{MT} + \frac{4e^{tfT}}{-1 + e^{2tfT}}, \\
d &= be^{MT} + c + 2\frac{1 + e^{2tfT}}{-1 + e^{2tfT}},
\end{aligned}
$$

where we have missed out common factors. Therefore,

$$a = (b - ce^{MT})M / N = \frac{H_0\{M(M - N) - e^{2MT}M(M + N)\} + H_T 2MNe^{MT}}{(M - N)^2 - e^{2MT}(M + N)^2}.$$

Notice that in a normal case we expect $M$ to be much bigger than $N$ and then, $a = H_0$. We now see if this is true in our case. Our total answer is,

$$a = \Sigma_F \int dk(\beta_{Fk}e^{-UT} + \gamma_{Fk}e^{UT})\frac{F}{4\pi^2 R}e^{ikz}\Sigma_f e^{ifx}\int d\zeta e^{i(F-f)\zeta}D[\zeta]$$
$$\times \int_{-\infty}^{+\infty} dK e^{iK(z-Z)}\theta_K(-it\Sigma^2)$$
$$\times \frac{H_0\{M(M-N) - e^{2MT}M(M+N)\} + H_T 2MNe^{MT}}{(M-N)^2 - e^{2MT}(M+N)^2},$$

where,

$$H_0 = 4e^{tfT}/(-1+e^{2tfT}) \qquad H_T = 2(1+e^{2tfT})/(-1+e^{2tfT})$$
$$M = (f^2 + K^2 + i\Sigma^2)^{1/2} \qquad N = (f^2 + K^2)^{1/2}$$
$$\theta_K = j\Sigma/(K^2 + i\Sigma^2).$$

This integral over $K$ is too difficult to perform but we notice that the $K$ complex plane has two poles at $K = \pm ij\Sigma$; and four branch points at $K = \pm if$ and $K = \pm i(f^2 + i\Sigma^2)^{1/2}$. Therefore we are obliged to make two cuts, one in the upper and one in the lower half plane. One cut is described by,

$$K = +i(f^2 + i\zeta\Sigma^2)^{1/2} \quad \text{where} \quad 0 < \zeta < 1.$$

The other cut is

$$K = -i(f^2 + i\zeta\Sigma^2)^{1/2} \quad \text{where} \quad 0 < \zeta < 1.$$

The contour must be closed in either the U.H.P. or the L.H.P. depending on the sign of $(z - Z)$ and can then be transformed into the sum of two parts, the pole at $\pm ij\Sigma$ and the difficult integral around the cut.

We first note that this result is symmetrical in $z - Z$. Next we note that at the pole $N$ has the value $(f^2 + K^2)^{1/2} = (f^2 - i\Sigma^2)^{1/2}$. This describes a signal decaying rapidly as we move away from the tube wall.

Along the cut,

$$N = (f^2 + K^2)^{1/2} = (f^2 - f^2 - i\zeta\Sigma^2)^{1/2} = ij\Sigma\zeta^{1/2},$$

so the integral over part of the cut (where $\zeta$ is small) produces a slowly decaying signal but the bulk of the integral has $N$ of the order of $\Sigma$ and therefore rapidly decays away from the tube wall.

We conclude that the final signal is difficult to observe unless the detector is close to the tube surface.We stress again the extreme rapidity of the signal decay; in free space, where there are no eddy currents, the decay takes place over a length comparable to the penetration depth in the tube wall itself. Therefore there may be merit to performing the entire examination using a LOWER-FREQUENCY, which automatically has a longer skin depth. Because $\Sigma$ varies like $\omega^{1/2}$, lowering the frequency by a factor of ten would extend the skin depth by a factor of 3.3. Instinctively this seems desirable.

We also ask what other information can be deduced? Let us estimate the total signal at a fixed distance from the tube wall. This is,

$$TOTAL = \int dza.$$

Now,

$$\int dze^{iK(z-Z)} = 2\pi\delta[K],$$

so,

$$\int dK \int dz e^{iK(z-Z)} F[K] = 2\pi F[0].$$

We note

$$\theta[0] = 1/(j\Sigma) \qquad N[0] = f \qquad M[0] = (f^2 + i\Sigma^2)^{1/2}.$$

Because $M[0]$ is so much larger than $N[0]$, we need retain only the terms proportional to $M^2$ in the fraction to give the simple result

$$TOTAL = \Sigma_F \int dk(\beta_{Fk} e^{-UT} + \gamma_{Fk} e^{UT}) \frac{F}{4\pi^2 R} e^{ikZ} \Sigma_f e^{ifx} \int d\zeta e^{i(F-f)\zeta} D[\zeta] \times (-jt\Sigma) \frac{4e^{tfT}}{-1 + e^{2tfT}}$$

This proves that the total signal is constant if measured over a large range of $z$, but that is difficult in most circumstances and impossible where the tube reaches a bend. This justifies the simple assumption made in the previous section. The final conclusion is that if the signal is hard to see, get closer to the tube wall or use a lower frequency.